STATIONARY MODE OF A NONLINEAR ELASTICALLY HEREDITARY OSCILLATOR

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The steady-state forced oscillations of a single-mass system subject to an external pure harmonic force are considered. The role of restoring force is played by a nonlinear function, which takes account of hereditary effects as a sum of multiple integrals in accordance with the Volterra theory. The problem is solved by the method of equivalent linearization, the discussion being confined to a triple integral of the hereditary type. The influence of the hereditary nonlinearity on the system dynamic characteristics, namely, its amplitude, phase, dynamic rigidity, hysteresis loop area, and Q, is investigated. In particular, the reciprocal of the Q, which can be taken as a measure of the internal friction, is shown to be independent of the amplitude of oscillation and to be the same as that obtained by linear theory. The other dynamic characteristics prove sensitive to the nonlinear properties. The exponential rational fractions proposed by Yu. N. Rabotnov are used as concrete hereditary functions.

1. In many physical phenomena the connection between the output signal x(t) and input signal y(t) is expressed by a relationship of the hereditary type, i.e., the system response at a given instant t is determined, not merely by the input signal at this instant, but also by the input excitation throughout the period prior to t. The Volterra nonlinear equations [1], connecting the input and output signals of a system invariant under changes in the time origin, can be written as [2]

$$x(t)=\sum_{m=1}^{\infty}\int_0^{\infty}\overset{m}{\cdots}\int f_m(t_1,t_2,\ldots t_m)\prod_{i=1}^{m}y(t-t_i)\,dt_i \tag{1.1}$$

$$y(t)=\sum_{n=1}^{\infty}\int_0^{\infty}\overset{n}{\cdots}\int g_n(t_1,t_2,\ldots t_n)\prod_{j=1}^{n}x(t-t_j)\,dt_j \tag{1.2}$$

Here, the functions $f_m(t_1, t_2, \ldots, t_m)$ take account of m-th order retardation effects, while the corresponding resolvents $g_n(t_1, t_2, \ldots, t_n)$ allow for n-th order relaxation effects; when $m = n = 1$, (1.1) and (1.2) become the Boltzmann linear hereditary equations. The lower limit of integration is zero, in accordance with the principle of causality, since the response x(t) cannot precede the input signal y(t).

In view of (1.1) and (1.2), there is a relationship between the functions $f_m(t_1, t_2, \ldots, t_m)$ and $g_n(t_1, t_2, \ldots, t_n)$, which can be conveniently written in Fourier space. In fact, on Fourier-transforming (1.1) and (1.2), we get

$$X(\omega)=\sum_{m=1}^{\infty}\int_0^{\infty}\overset{(m)}{\cdots}\int F_m(\omega_1,\ldots,\omega_m)\,\delta\Big(-\omega+\sum_{i=1}^{m}\omega_i\Big)\prod_{k=1}^{m}Y(\omega_k)\,d\omega_k \tag{1.3}$$

$$Y(\omega)=\sum_{n=1}^{\infty}\int_0^{\infty}\overset{(n)}{\cdots}\int G_n(\omega_1',\ldots,\omega_n')\,\delta\Big(-\omega+\sum_{j=1}^{n}\omega_j'\Big)\prod_{l=1}^{n}X(\omega_l')\,d\omega_l' \tag{1.4}$$

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Here, the upper case letter denotes the Fourier integral transform of the corresponding lower case letter:

$$X(\omega) = (2\pi)^{-1/2}\int_0^\infty x(t)\,e^{-i\omega t}dt, \quad Y(\omega) = (2\pi)^{-1/2}\int_0^\infty y(t)\,e^{-i\omega t}dt \tag{1.5}$$

$$G_n(\omega_1, \ldots, \omega_n) = (2\pi)^{-1/2(n-1)}\int_0^\infty \overset{n}{\cdots}\int \exp\left(-i\sum_{\nu=1}^{n}\omega_\nu t_\nu\right) g_n(t_1, \ldots, t_n)\,dt_1 \ldots dt_n \tag{1.6}$$

An expression similar to (1.6) can be written for $F_m(\omega_1, \ldots, \omega_m)$.

Substitution of (1.4) in (1.3) gives

$$X(\omega) \equiv \sum_{m=1}^{\infty}\int_0^\infty \overset{m}{\cdots}\int F_m(\omega_1, \ldots, \omega_m)\,\delta\left(-\omega + \sum_{i=1}^{m}\omega_i\right)\prod_{k=1}^{m}$$
$$\times\left\{\sum_{n=1}^{\infty}\int_0^\infty \overset{n}{\cdots}\int G_n(\omega_1', \ldots, \omega_n')\,\delta\left(-\omega_k + \sum_{j=1}^{n}\omega_j'\right)\prod_{l=1}^{n} X(\omega_l')\,d\omega_l'\right\} \tag{1.7}$$

This identity enables a connection to be established between the functions F_m and G_n for any values of m and n. To obtain this connection, concrete values have to be assigned to the indices when writing out the right side of (1.7); then the coefficient of the linear term in $X(\omega)$ is put equal to unity, while each nonlinear term in $X(\omega)$ has to vanish. Inversion equations are thus obtained for each value of m and n; due to their complexity, the formulas are usually only quoted for the first three values:

$$F_1(\omega)G_1(\omega) = 1, \qquad G_2(\omega_1,\omega_2) = -F_2(\omega_1, \omega_2)[F_1(\omega_1)F_2(\omega_2)F_1(\omega_1+\omega_2)]^{-1}$$
$$G_3(\omega_1, \omega_2, \omega_3) = \left[\prod_{k=1}^{3} F_1(\omega_k)\,F(\omega_1+\omega_2+\omega_3)\right]^{-1}\left\{-F_3(\omega_1, \omega_2, \omega_3)\right.$$
$$\left.+\frac{2}{3}\sum_{i\to j\to k\to i}[F_1(\omega_j+\omega_k)]^{-1}F_2(\omega_j, \omega_k)\,F_2(\omega_i, \omega_j+\omega_k)\right\} \tag{1.8}$$

By this means, either (1.1) or (1.2) may be used, according to the concrete problem.

2. Consider the stationary mode of a one-dimensional oscillator moving under the action of an external pure harmonic force $P\cos\omega t$. Assuming that $x(t)$ represents the displacement, and $y(t)$ the restoring force, the equation of motion becomes

$$Mx^{\cdot\cdot} + y(x,x^{\cdot}) = P\cos\omega t \tag{2.1}$$

where M is the mass and the dot denotes differentiation with respect to time.

The solution of (2.1) will be found by the Krylov-Bogolyubov method of equivalent linearization [3]. The equation is first rewritten as

$$Mx^{\cdot\cdot} + \omega^{-1}\eta x^{\cdot} + kx + \varepsilon(x,x^{\cdot}) = P\cos\omega t \tag{2.2}$$

where $\varepsilon(x, x^{\cdot})$ denotes the error that results from replacing the nonlinear function $y(x, x^{\cdot})$ by an equivalent linear visco-elastic part, i.e.,

$$\varepsilon(x,x^{\cdot}) = y(x,x^{\cdot}) - kx - \omega^{-1}\eta x^{\cdot} \tag{2.3}$$

The stationary solution of (2.2), with $\varepsilon(x, x^{\cdot}) = 0$, is

$$x = A\cos\theta, \quad \theta = \omega t - \varphi \tag{2.4}$$

whence the amplitude A and tangent of the phase-shift are easily found:

$$A = P[\eta^2 + (k - M\omega^2)^2]^{-1/2}, \quad \operatorname{tg}\varphi = \eta(k - M\omega^2)^{-1} \tag{2.5}$$

The coefficients k and η are found from the minimization condition for the error $\varepsilon(x, x^{\cdot})$, which is written in the form of two equations, averaged over the period of oscillation $T = 2\pi\omega^{-1}$ [4, 5]:

$$\left\langle\frac{\partial}{\partial\eta}[\varepsilon(x, x^{\cdot})]^2\right\rangle = 0, \qquad \left\langle\frac{\partial}{\partial k}[\varepsilon(x, x^{\cdot})]^2\right\rangle = 0 \tag{2.6}$$

On substituting (2.3) in (2.6), expressions are obtained for η and k that allow for the nonlinear properties of the system:

$$k = (\pi A)^{-1}\int_0^{2\pi} y(A, \theta)\cos\theta\, d\theta, \quad \eta = -(\pi A\omega)^{-1}\int_0^{2\pi} y(A, \theta)\sin\theta\, d\theta \tag{2.7}$$

The role of dynamic modulus of the system is played by k, while η is proportional to the hysteresis loop area. The reciprocal of the system Q is taken as a measure of the internal friction, and is given by

$$Q^{-1} = \frac{1}{2\pi}\frac{\Delta W}{W} = \frac{P}{\pi k A^2}\int_0^T x^{\cdot}(t)\cos\omega t dt = \frac{P\sin\varphi}{Ak} = \frac{\eta}{k}. \tag{2.8}$$

Notice that Q^{-1} is the same as the tangent of the phase-shift tg φ in the quasi-static case, i.e., when M = 0.

3. The method described in Section 2 can be applied to the system (1.2). Substituting (1.2) in (2.7) and recalling (2.4), it is found that the coefficients k and η are determined solely by the odd terms in (1.2). Retaining only the first three terms in (1.2), the following expressions are obtained:

$$k = \mathrm{Re}\ G_1(\omega) + {}^3/_2\pi A^2 \mathrm{Re}\, G_3(\omega, \ \omega, -\omega) \tag{3.1}$$

$$-\eta = \mathrm{Im}\ G_1(\omega) + {}^3/_2\ \pi A^2 \mathrm{Im}\ G_3(\omega, \ \omega, -\omega) \tag{3.2}$$

Here, Re G_n and Im G_n are the real and imaginary parts of the complex quantity G_n given by (1.6). The amplitude A is found as a function of the frequency and rheological parameters of the system from a sixth degree equation, obtained by substituting (3.1) and (3.2) in the first of (2.5):

$$\begin{gathered} aA^6 + bA^4 + cA^2 + d = 0 \\ a = ({}^3/_2\pi)^2[(\mathrm{Re}\, G_3)^2 + (\mathrm{Im}\, G_3)^2], \quad c = (\mathrm{Im}\, G_1)^2 + [\mathrm{Re} G_1 - M\omega^2]^2 \\ b = 3\pi[\mathrm{Re}\, G_1\ \mathrm{Im}\ G_3 + \mathrm{Re}\, G_3(\mathrm{Re}\, G_1 - M\omega^2)], \qquad d = -P^2 \end{gathered} \tag{3.3}$$

It is easily seen that the coefficient c is equal to the square of the reciprocal of the amplitude for a linear oscillator, i.e., in the case when a = b = 0 or Re G_0 = Im G_3 = 0. In this case the system dynamic characteristics, namely, k, η, tg φ, and Q^{-1}, are independent of the amplitude; in the nonlinear case, they are functions of the amplitude A, which is given by the complicated equation (3.3).

Solution of the problem can be somewhat simplified if the functions $g_1(t_1)$ and $g_3(t', t_2, t_3)$ are specially chosen as follows [6, 7]:

$$g_1(t_1) = g(t_1), \ \ g(t_1, \ t_2, \ t_3) = g(t_1)g(t_2)g(t_3) \tag{3.4}$$

Then,

$$\mathrm{Re}\, G_3 = (2\pi)^{-1}G_*\mathrm{Re} G, \quad \mathrm{Im}\, G_3 = (2\pi)^{-1}G_*\mathrm{Im}\, G, \quad G_* = (\mathrm{Re}\, G)^2 + (\mathrm{Im}\, G)^2 \tag{3.5}$$

$$a = ({}^3/_4)^2 G_*^3, \ b = {}^3/_2 G_*(G_* - M\omega^2 \mathrm{Re}\, G), \ c = G_* - 2M\omega^2 \mathrm{Re}\, G + M^2\omega^4 \tag{3.6}$$

$$k = (1 + {}^3/_4 A^2 G_*)\mathrm{Re}\, G, \ -\eta = (1 + {}^3/_4 A^2 G_*)\ \mathrm{Im}\ \ G \tag{3.7}$$

It immediately follows from this that the internal friction Q^{-1} is independent of the oscillation amplitude. For, substituting (3.7) in (2.8), Q^{-1} is found to be given by just the first term of (2.1), i.e., is the same as the value obtained from linear theory:

$$Q^{-1} = -\mathrm{Im}\ G(\omega)\ [\mathrm{Re}\, G(\omega)]^{-1}, \ \mathrm{Im}\, G(\omega) < 0 \tag{3.8}$$

4. To investigate the influence of the nonlinear properties on the amplitude A and the other dynamic characteristics, g(t) will be written as

$$\begin{gathered} g(t) = k_\infty[\delta(t) - \nu_\varepsilon \tau_\varepsilon^{-\gamma} \mathrm{E}_\gamma(-1, t, \tau_\varepsilon)], \\ \mathrm{E}_\gamma(-1, t, \tau_\varepsilon) \equiv t^{\gamma-1}\sum_{n=0}^{\infty}\frac{(-1)^n (t/\tau_\varepsilon)^{\gamma n}}{\Gamma[\gamma(n+1)]} \end{gathered} \tag{4.1}$$

where $\delta(t)$ is the Diract δ-function, $\mathrm{E}_\gamma(-1, t, \tau_\varepsilon)$ is the Rabotnov exponential rational fraction [8], $\gamma \in (0, 1)$ is the quality parameter, τ_ε the relaxation time, k_∞ and k_0 are respectively the unrelaxed and relaxed values of the elastic modulus, and ν_ε, ν_σ are the modulus deficiencies:

$$\nu_\varepsilon = (k_\infty - k_0) k_\infty^{-1}, \qquad \nu_\sigma = (k_\infty - k_0) k_0^{-1}$$

Using (4.1), the following expressions are easily obtained for the real and imaginary parts of $G(\omega)$ in the Fourier space:

$$\operatorname{Re} G = k_\infty\left[1 - \nu_\varepsilon \frac{\varkappa^{-\gamma} + \cos\psi}{\Omega(1,1)}\right], \qquad \operatorname{Im} G = -\nu_\varepsilon k_\infty \frac{\sin\psi}{\Omega(1,1)}$$
$$\Omega(u, v) = u^2\varkappa^\gamma + v^2\varkappa^{-\gamma} + 2uv\cos\psi, \qquad \varkappa = \omega\tau_\varepsilon, \quad \psi = {}^1/_2\pi\gamma \tag{4.2}$$

Substitution of (4.2) in (3.5), (3.6), and (3.7) gives

$$a = \frac{9}{16}\left[\frac{\Omega(k_\infty, k_0)}{\Omega(1,1)}\right]^3, \qquad c = \frac{\Omega(k_\infty - M\omega^2, k_0 - M\omega^2)}{\Omega(1,1)}$$
$$b = \frac{3}{2}\left[\frac{\Omega(k_\infty, k_0)}{\Omega(1,1)}\right]^2\left\{1 - \frac{M\omega\,[k_\infty\varkappa^\gamma + k_0\varkappa^{-\gamma} + (k_\infty + k_0)\cos\psi]}{\Omega(k_\infty, k_0)}\right\} \tag{4.3}$$

$$k = k_\infty\left[1 - \nu_\varepsilon\frac{\varkappa^{-\gamma} + \cos\psi}{\Omega(1,1)}\right]\left[1 + \frac{3}{4}\frac{A^2\Omega(k_\infty, k_0)}{\Omega(1,1)}\right]$$
$$\eta = (k_\infty - k_0)\frac{\sin\psi}{\Omega(1,1)}\left[1 + \frac{3}{4}\frac{A^2\Omega(k_\infty, k_0)}{\Omega(1,1)}\right] \tag{4.4}$$

The behavior of the amplitude A as a function of the frequency ω and the rheological parameters was examined in [9, 10] in the linear case, where $A = 1/\sqrt{c}$. The main features of the frequency-dependence of A in the nonlinear case can be seen without performing the laborious computations needed for solving the sixth degree Eq. (3.3), by confining the discussion to two limiting elastic cases, namely, the relaxed and the unrelaxed. It is assumed, in fact, that the amplitude A will vary slowly at either high or low frequencies, and asymptotic formulas are written for k and η in the respective cases $\varkappa \gg 1$ and $\varkappa \ll 1$. There is an essential difference here between the cases $\gamma \neq 1$ and $\gamma = 1$:

$$\gamma \neq 1, \varkappa \gg 1, \quad k \approx k_\infty[1 - \nu_\varepsilon\varkappa^{-\gamma}\cos\psi + {}^3/_4 A^2 k_\infty^2(1 - 3\nu_\varepsilon\varkappa^{-\gamma}\cos\psi)]$$
$$\eta \approx (k_\infty - k_0)(1 + {}^3/_4 A^2 k_\infty^2)\varkappa^{-\gamma}\sin\psi \tag{4.5}$$

$$\varkappa \ll 1, \quad k \approx k_0[1 + \nu_\sigma\varkappa^\gamma\cos\psi + {}^3/_4 A^2 k_0^2(1 + 3\nu_\sigma\varkappa^\gamma\cos\psi)]$$
$$\eta \approx (k_\infty - k_0)(1 + {}^3/_4 A^2 k_0^2)\varkappa^\gamma\sin\psi \tag{4.6}$$

$$\gamma = 1, \varkappa \gg 1, \quad k \approx k_\infty\{1 - \nu_\varepsilon\varkappa^{-2} + {}^3/_4 A^2 k_\infty^2[1 - \nu_\varepsilon\varkappa^{-2}(2 + k_0 k_\infty^{-1})]\}$$
$$\eta \approx (k_\infty - k_0)(1 + {}^3/_4 A^2 k_\infty^2)\varkappa^{-1} \tag{4.7}$$

$$\varkappa \ll 1, \quad k \approx k_0\{1 + \nu_\sigma\varkappa^2 + {}^3/_4 A^2 k_0^2[1 + \nu_\sigma\varkappa^2(2 + k_\infty k_0^{-1})]\}$$
$$\eta \approx (k_\infty - k_0)(1 + {}^3/_4 A^2 k_0^2)\varkappa \tag{4.8}$$

On taking the limits of k and η from (4.5)-(4.8) as $\tau_\varepsilon \to \infty$ and $\tau_\varepsilon \to 0$, and substituting in the first of (2.5), then solving the latter for the delay term $M\omega^2$, two relationships are obtained, defining the frequency curves for the two amplitudes corresponding to relaxed and unrelaxed oscillations:

$$\omega^2 = \omega_0^2(1 + {}^3/_4 k_0^2 A^2) \pm P(MA)^{-1}, \qquad \omega_0^2 = k_0 M^{-1}$$
$$\omega^2 = \omega_\infty^2(1 + {}^3/_4 k_\infty^2 A^2) \pm P(MA)^{-1}, \qquad \omega_\infty^2 = k_\infty M^{-1} \tag{4.9}$$

These expressions describe the oscillatory processes for systems with a rigid response, similar to the processes discussed, e.g., in [11]. Figure 1 gives $A = A(\omega^2)$ curves for the numerical data

$$\omega_\infty^2 = 1, \; k_\infty^2 = {}^4/_3, \; k_0^2 = {}^1/_3,$$
$$\nu_\varepsilon = {}^1/_2, \; PM^{-1} = 1$$

and it can be seen that there are two elastic resonances, at $\omega_\infty^2 = 1$ and $\omega_0^2 = 1/2$. When the relaxation time τ_ε has a finite value, the resonant amplitudes will occupy an intermediate position; and here, as in the linear case, when $\gamma = 1$ all the resonant amplitudes will have a common point of intersection [7, 10]. The difference lies in the fact that, whereas in the linear case the frequency ω_* at which the resonant amplitudes intersect is determined by the modulus deficiency, or by

$$\omega_*^2 = {}^1/_2(\omega_\infty^2 + \omega_0^2) \tag{4.10}$$

in the nonlinear case ω_* is amplitude-dependent and given by

$$\omega_*^2 = \tfrac{1}{2}(\omega_\infty^2 + \omega_0^2) + \tfrac{3}{8}A^2(k_\infty^2\omega_\infty^2 + k_0^2\omega_0^2) \tag{4.11}$$

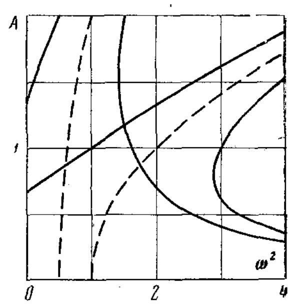

Fig. 1

When considering the tangent of the phase-shift (tg φ), its dependences on the relaxation time τ_ε and on the frequency ω need to be considered separately. The asymptotic equations will first be written for large and small values of τ_ε, retraining only first order terms:

$$\tau_\varepsilon \gg 1, \quad \mathrm{tg}\,\varphi \approx (\omega_\infty^2 - \omega_0^2)(1 + \tfrac{3}{4}k_\infty^2A^2)(\omega_\infty^2 - \omega^2 + \tfrac{3}{4}k_\infty^2A^2)^{-1}\varkappa^{-\gamma}\sin\psi \tag{4.12}$$

$$\tau_\varepsilon \ll 1, \quad \mathrm{tg}\,\varphi \approx (\omega_\infty^2 - \omega_0^2)(1 + \tfrac{3}{4}k_0^2A^2)(\omega_0^2 - \omega^2 + \tfrac{3}{4}k_0^2A^2)^{-1}\varkappa^{\gamma}\sin\psi \tag{4.13}$$

It can be seen from these expressions that tg $\varphi = 0$ for the limiting elastic cases.

Next, it is easily seen from (2.5) and (4.4) that, at low frequencies ($\omega \ll 1$), tg $\varphi \approx Q^{-1}$ and is independent of the amplitude. At high frequencies ($\omega \gg 1$), tg $\varphi \to 0$ from negative values, i.e., the angle $\varphi \to \pi$. A similar picture is obtained in the linear case [7, 10]. Nonlinear behavior of tg φ, leading to a shift of the φ curves in the direction of increasing A as the amplitude rises, occurs in the intermediate frequency range and is most pronounded when $\omega \sim 1$.

To sum up, investigation of the stationary mode of a nonlinear elastically hereditary oscillator reveals how the nonlinear properties influence the system dynamic characteristics. In particular, Q^{-1}, of the internal friction, proves to be independent of the amplitude of oscillation, i.e., of the nonlinear properties. This fact needs to be borne in mind when examining retardation–relaxation processes by the method of internal friction, since the temperature-frequency behavior of Q^{-1} fails to provide complete information on the properties of the elastic material. In view of this, in addition to the dissipative response of Q^{-1} we need to examine the behavior of the other dynamic characteristics, and notably, the resonant amplitude. This point is specially important when it comes to considering the so-called "background" of internal friction, concerning the nature of which there are still contradictory opinions.

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